Copula Theory And Its Applications

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Copula Definition

A two-dimensional **copula** is a function C whose domain is $\mathbb{I}^2$ and range is $\mathbb{I}$ and has the following properties:

1. C is **grounded**: $C(u,0) = 0 = C(0,v)$, $\forall u, v \in \mathbb{I}$.
2. $C(u,1) = u$ and $C(1,v) = v$, $\forall u, v \in \mathbb{I}$.
3. C is **2-increasing**: $\forall u_1, u_2, v_1, v_2 \in \mathbb{I}$ such that $u_1 \le u_2$ and $v_1 \le v_2$,

$V_C([u_1,u_2] \times [v_1,v_2]) = C(u_2,v_2) - C(u_2,v_1) - C(u_1,v_2) + C(u_1,v_1) \ge 0$

Sklar' Theorem

Let H be a joint distribution function with margins F and G. Then, there exists a copula C such that $\forall x, y \in \overline{\mathbb{R}}$,

$$H(x,y) = C(F(x), G(y))$$

Conversely, if C is a copula and F and G are distribution functions, then the function H defined by (**??**) is a joint distribution function with margins F and G.

Transformation Property

Let X and Y be continuous random variables with copula C_{XY}. Let α and β be strictly monotone on $RanX$ and $RanY$, respectively.
1. If α and β are both strictly increasing, then

$$C_{\alpha(X)\beta(Y)}(u,v) = C_{XY}(u,v).$$

Thus, C_{XY} is invariant under strictly increasing transformations of X and Y.
2. If α is strictly increasing and β are strictly decreasing, then

$$C_{\alpha(X)\beta(Y)}(u,v) = u - C_{XY}(u, 1-v).$$

3. If α and β are both strictly decreasing, then

$$C_{\alpha(X)\beta(Y)}(u,v) = u + v - 1 + C_{XY}(1-u, 1-v).$$

Measures Of Assocation

Kendall's tau: Let (X_1, Y_1) and (X_2, Y_2) be independent and identically distributed random vectors, each with joint distribution function H. Then the population version of Kendall's tau is defined as the probability of concordance minus the probability of discordance:

$$\tau = \tau_{X,Y} = P[(X_1 - X_2)(X_1 - X_2) > 0] - P[(X_1 - X_2)(X_1 - X_2) < 0]$$

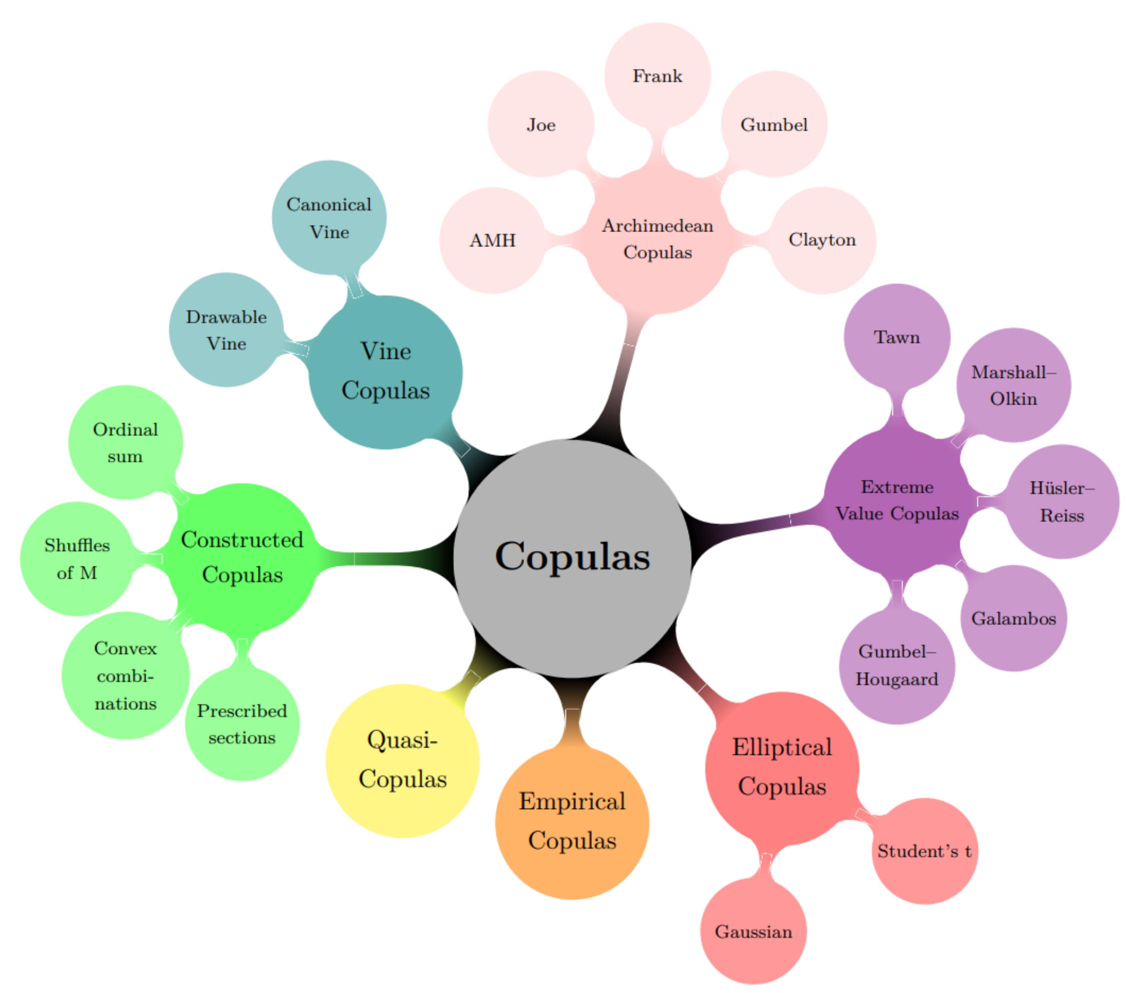
$$\tau_{X,Y} = \tau_C = Q(C,C) = 4 \iint_{\mathbb{I}^2} C(u,v)\, dC(u,v) - 1$$

Spearman's rho: Let (X_1, Y_1), (X_2, Y_2), and (X_3, Y_3) be three independent random vectors with a common joint distribution function H (whose margins are again F and G) and copula C. The population version $\rho_{X,Y}$, of Spearman's rho is defined to be proportional to the probability of concordance minus the probability of discordance for the two vectors (X_1, Y_1) and (X_2, Y_3):

$$\rho_{X,Y} = 3(P[(X_1 - X_2)(Y_1 - Y_3) > 0] - P[(X_1 - X_2)(Y_1 - Y_3) < 0]$$

$$\rho_{X,Y} = \rho_C = 3Q(C, \Pi) = 12 \iint_{\mathbb{I}^2} C(u,v) dudv - 3.$$

Types Of Copulas



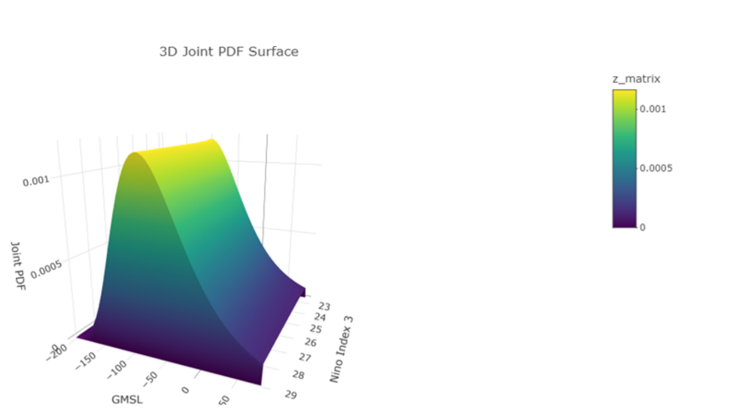
Practical Application

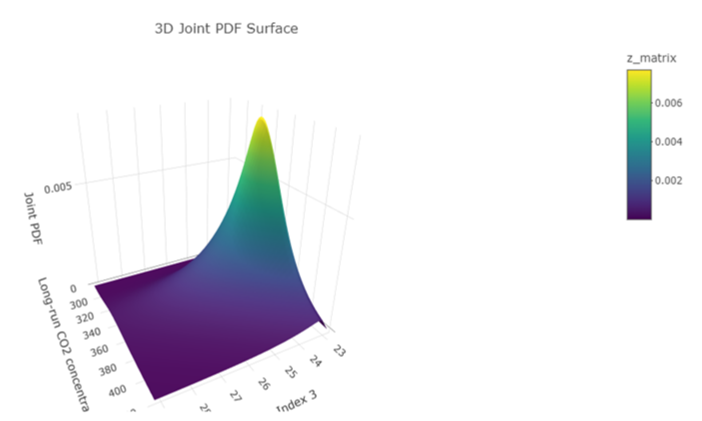
Objective: The primary objective of our analysis was to investigate the dependence structure between the Niño 3 region temperature (1880–2024) and key climate indicators that are potentially interlinked with it. These indicators included Global Mean Sea Level (GMSL), long-run atmospheric concentrations of CO_2 and N_2O, and the average extent of Arctic and Antarctic Sea Ice.

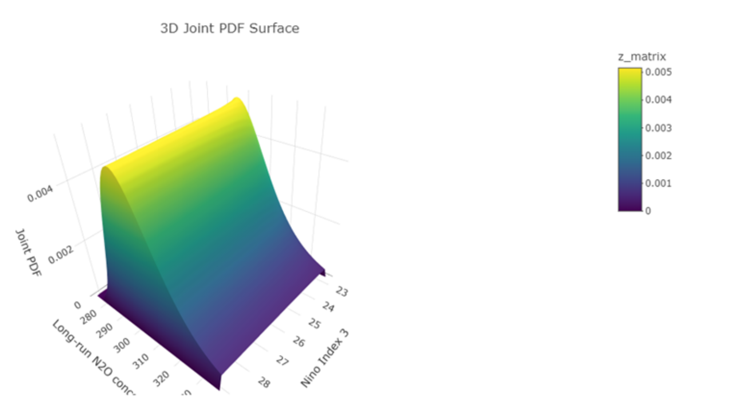
Procedure: First, we fitted appropriate univariate distributions to each of the six datasets to capture their individual characteristics. Next, we applied bivariate copula models to each pair consisting of Niño 3 temperature and one of the other five variables. For model selection, we used the Kolmogorov–Smirnov (KS) test to evaluate the goodness-of-fit and determine the most suitable copula family for each relationship. This approach enabled us to explore non-linear and tail dependencies beyond what traditional correlation measures can capture.

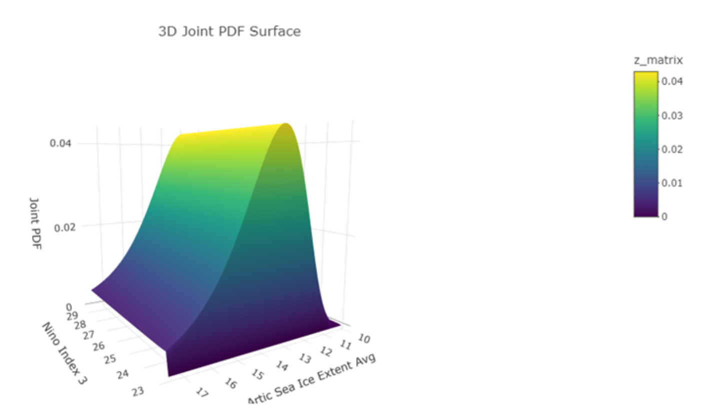
Factor	AMH	Joe	Frank	Gumbel	Clayton	Gaussian	Student's t	Parameter	(τ)	(ρ)
GMSL	0.869	0.948	0.981	0.896	**0.981**	0.943	0.930	0.070	0.034	0.051
CO2	0.869	0.967	0.968	0.878	**0.982**	0.921	0.933	0.059	0.029	0.044
N2O	0.869	0.973	0.976	0.867	**0.981**	0.943	0.933	0.065	0.032	0.047
Arctic	0.704	0.956	0.976	0.98	**0.984**	0.930	0.916	-0.024	-0.030	-0.045
Antarctica	0.742	0.956	0.982	0.984	**0.984**	0.912	0.911	-0.025	-0.021	-0.032

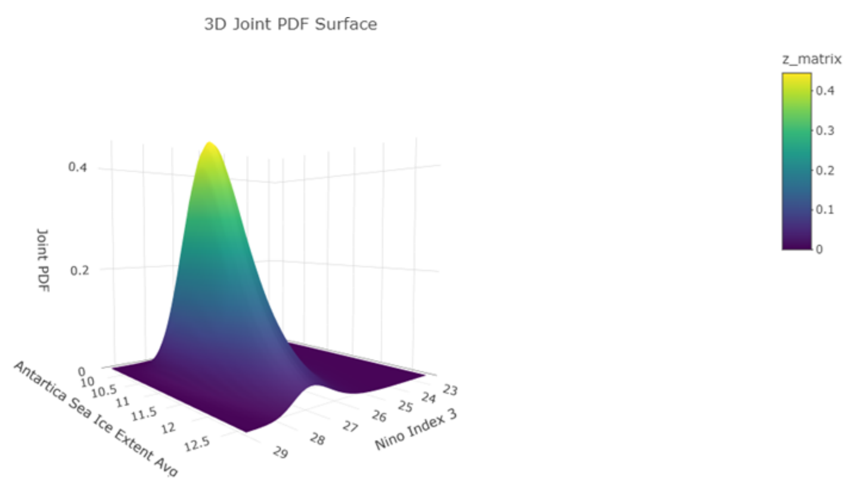
Table 1. Nino 3 temperature: p-values of Copulas, Kendall's tau, Spearman's rho











Future Research Prospects

Further study of the use of Copula Functions in the field of Stochastic Processes, Actuarial Science, and Machine Learning is to be done.

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