Serre Duality on Complex Manifolds

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The Serre Duality is a classical equivalence between the cohomology group of the sheaf of holomorphic forms over a holomorphic vector bundle E on a compact complex manifold X and the associated cohomology group over the dual bundle E'. First stated and proved by Jean Pierre Serre and extended by Grothendieck, this duality theorem is ubiquitous in Algebraic Geometry. To see this duality in its entirety, we assume the following notation throughout:

Notation

- $\varepsilon^r(X) = \varepsilon^r(X, \wedge^r T^*(X) \otimes \mathbb{C})$, the sheaf of sections of complexified $r-$ differential forms on X.
- $\varepsilon^{p;q}(X) = \varepsilon^{p;q}(X, \wedge^{p;q} T^*(X) \otimes \mathbb{C})$, associated $(p,q)-$forms, based on the decomposition of the complex structure:
$$\varepsilon^r(X) = \oplus_{p+q=r} \varepsilon^{p;q}(X)$$
- $\varepsilon^r(X, E) = \varepsilon^r(X, \wedge^r T^*(X)_c \otimes E)$, the sheaf of sections of complexified $r-$ differential forms on X, with coefficients in a holomorphic bundle E.
- $\varepsilon^{p;q}(X) = \varepsilon^r(X, \wedge^{p;q} T^*(X)_c \otimes E)$, associated $(p,q)-$forms.

The Hodge-* operator

The Hodge-* operator is:

- On general vector spaces V of dimension n with an orthonormal basis $\{e_1, \dots e_n\}$:
$$* : \wedge^k V \to \wedge^{n-k} V \text{ is defined by } * e_{i_1} \wedge \ldots \wedge e_{i_k} = \pm e_{j_1} \wedge \ldots \wedge e_{j_{n-k}}$$
where $\{j_1, \dots, j_{n-k}\} = \{i_1, \dots, i_k\}^c$, based on an oriented volume form $e_1 \wedge \ldots \wedge e_n = \mathcal{V}_f$.
- For $\alpha, \beta \in \wedge^k V$,
$$\alpha \wedge *\beta = \sum_{|I|=k}{}' \alpha_I \beta_I \mathcal{V}_f.$$

This gives an associated inner product on $\wedge^k V$ induced by the inner product on V, given by:

$$\langle \alpha, \beta \rangle = \sum_{|I|=k}{}' \alpha_I \beta_I.$$

- On differential $(p,q)-$forms on a complex manifold X of dimension r:
$$* : \varepsilon^{p;q}(X) \to \varepsilon^{n-p;n-q}(X) \text{ defined by}$$
$$f(x) = \sum_{|I|=p,|J|=q}{}' f_{IJ}(x) dz_I \wedge d\overline{z_J} \implies *f(x) := \sum_{|I^c|=n-p,|J^c|=n-q}{}' f_{IJ}(x) dz_{I^c} \wedge d\overline{z_{J^c}}.$$
- On differential forms on a complex manifold X with coefficients in a holomorphic bundle $\pi : E \to X$:
$$\overline{*}_E : \wedge^r T^*(X)_c \otimes E \to \wedge^{2n-r} T^*(X)_c \otimes E' \text{ defined by}$$
$$\overline{*}_E(\alpha \otimes e) := \overline{*}(\alpha) \otimes \tau(e),$$
where $\tau : E \to E'$ is an isomorphism of bundles.

Result 1

$\varepsilon^{p;q}(X, E) \xrightarrow{\overline{*}_E} \varepsilon^{n-p;n-q}(X, E')$ is an isomorphism.

Harmonic theory on complex manifolds

Harmonic theory aims to define the adjoint of the d operator and its complex analog $\overline{\partial}$. We denote the adjoint as d^* and $\overline{\partial}^*$, respectively. The construction of these adjoint operators is as follows.

- Since adjoints need an inner product, we first define the **Hodge inner product** as $(.,.) : \varepsilon^\bullet(X, \wedge^\bullet T^*(X)) \times \varepsilon^\bullet(X, \wedge^\bullet T^*(X)) \to \mathbb{R}$ by:
$$(\phi, \psi) = \begin{cases} \int_X \phi \wedge \overline{*}\psi, \text{ if } \phi, \psi \in \varepsilon^p(X) \\ 0, \text{ if } \phi \in \varepsilon^p(X), \psi \in \varepsilon^q(X) \text{ with } p \neq q. \end{cases}$$
- **Form of the d^* operator:** For $r < n$, consider $\phi \in \varepsilon^{n-1}(X)$ and $\psi \in \varepsilon^r(X)$. Stokes' theorem gives us:
$$(d\phi, \psi) = (-1)^r \int_X \phi \wedge d\overline{*}\psi = (-1)^{r(1-n+r)} \int_X \phi \wedge \overline{*}(\overline{*} d \overline{*} \psi) = (-1)^{r(1-n+r)} (\phi, d^* \psi).$$
This gives:
$$d^* = (-1)^{r(1-n+r)} \overline{*} d \overline{*}.$$
We can then define the **Laplacian** as $\Delta := dd^* + d^*d$.
- **Hodge inner product on a holomorphic bundle E**: Likewise define
$$(\phi, \psi)_E := \begin{cases} \int_X \phi \wedge \overline{*}_E \psi \text{ if } \phi, \psi \in \varepsilon^p(X, E), \\ 0, \text{ if }, \phi \in \varepsilon^p(X, E), \psi \in \varepsilon^q(X, E), \text{ with } p \neq q. \end{cases}$$
- **Form of the $\overline{\partial}^*$ operator:** A similar application of Stokes' theorem and the general Hodge inner product gives
$$\overline{\partial}^* = -\overline{*}_{E^*} \overline{\partial} \overline{*}_E.$$
We can now define **the complex Laplacian** as $\overline{\square} := \overline{\partial}.\overline{\partial}^* + \overline{\partial}^*.\overline{\partial}$.
- $\overline{\square}$ orthogonally decomposes $\varepsilon^{p;q}(X, E)$ into:
$$\varepsilon^{p;q}(X, E) = \mathcal{H}^{p;q}(X, E) \oplus \mathcal{H}^{p;q}(X, E)^\perp$$
where $\mathcal{H}^{p;q}(X, E) = \{\xi \in \varepsilon^{p;q}(X, E) : \overline{\square}\xi = 0\}$ is termed as the **Harmonic subspace** and $\mathcal{H}^{p;q}(X, E)^\perp = \{f \in \varepsilon^{p;q}(X, E) : (f, \xi) = 0, \xi \in \mathcal{H}^{p;q}\}$ is its orthogonal complement.

An important projection

Denote the orthogonal projection of $\varepsilon^{p,q}(X, E)$ onto $\mathcal{H}^{p,q}(X, E)$ by $H_{\overline{\square}}$.

References

- *Differential Analysis on Complex Manifolds, Raymond Wells, Springer.*
- *Lectures on Compact Riemann Surfaces, Otto Forster, Springer.*
- *Differential Geometry of Complex Vector Bundles, Shoshichi Kobayashi.*
- *Introduction to Manifolds, Loring Tu.*

Dolbeault cohomology

The Dolbeault cohomology is the equivalence of $(p,q)-$ non exactness and the q^{th} cohomology group with coefficients in the sheaf of p- holomorphic forms over a holomorphic bundle E. Specifically, define:

$$H^{p,q}(X, E) := \frac{\text{Ker}(\varepsilon^{p,q}(X, E) \xrightarrow{\overline{\partial}} \varepsilon^{p,q+1}(X, E))}{\text{Im}(\varepsilon^{p,q-1}(X, E) \xrightarrow{\overline{\partial}} \varepsilon^{p,q}(X, E))}$$

to be the non-exactness of the short sequence:

$$0 \to X \hookrightarrow \Omega^p(X, E) \xrightarrow{\overline{\partial}} \varepsilon^{p;1}(X, E) \ldots \to \varepsilon^{p;n}(X, E) \to 0.$$

We first have a general construction:

Constructing long exact cohomological sequences

Consider the short exact sequence of sheaves:

$$0 \to \mathcal{F}_1 \xrightarrow{\alpha} \mathcal{F}_2 \xrightarrow{\beta} \mathcal{F}_3 \to 0.$$

where the exactness of sheaves means $\text{Ker}(\beta) = \text{Im}(\alpha)$ at the stalk $0 \to \mathcal{F}_{1x} \xrightarrow{\alpha} \mathcal{F}_{2x} \xrightarrow{\beta} \mathcal{F}_{3x} \to 0$ corresponding to each $x \in X$. A cohomological long-exact sequence can now be constructed:

$$\cdots H^p(X, \mathcal{F}_1) \xrightarrow{\alpha^*} H^p(X, \mathcal{F}_2) \xrightarrow{\beta^*} H^p(X, \mathcal{F}_3) \xrightarrow{\delta^*} H^{p+1}(X, \mathcal{F}_1) \xrightarrow{\alpha^*} H^{p+1}(X, \mathcal{F}_2) \xrightarrow{\beta^*} H^{p+1}(X, \mathcal{F}_3) \cdots$$

by defining $\alpha^*([f]) = [\alpha \circ f]$ and $\delta^*([c]) = [a]$, where a is a unique element in the cochain group $C^{p+1}(U, \mathcal{F}_1)$ such that $\alpha(a) = \delta(b)$, and b is obtained via the surjection $\beta(b) = c$.

Denoting $Z^{p,q}(X, E) = Ker(\varepsilon^{p,q}(X, E) \xrightarrow{\overline{\partial}} \varepsilon^{p,q+1}(X, E))$, this construction is applied to the following system of short exact chains:

$$0 \longrightarrow \underbrace{\Omega^p}_{H^q(X,\Omega^p)} \xrightarrow{i} \varepsilon^{p,0} \xrightarrow{\overline{\partial}} \overbrace{Z^{p,1}}^{H^{q-1}(X,Z^{p,1})} \longrightarrow 0$$

$$0 \longrightarrow Z^{p,1} \xrightarrow{i} \varepsilon^{p,1} \xrightarrow{\overline{\partial}} \overbrace{Z^{p,2}}^{H^{q-2}(X,Z^{p,2})} \longrightarrow 0$$

$$\ldots$$

$$0 \longrightarrow \overbrace{Z^{p,q}}^{H^0(X,Z^{p,q})} \xrightarrow{i} \varepsilon^{p,q} \xrightarrow{\overline{\partial}} Z^{p,q+1} \longrightarrow 0.$$

The cohomology groups highlighted in blue are 'chased' backward through the δ^* operator, giving:

$$H^q(X, \Omega^p) \cong H^{q-1}(X, Z^{p,1}) \cong \ldots \cong H^1(X, Z^{p,q-1}) \cong H^0(X, Z^{p,q}) / \overline{\partial} H^0(X, Z^{p,q-1}) \cong H^{p,q}.$$

Hence, we have:

Dolbeault's theorem

$$H^q(X, \Omega^p(E)) \cong H^{p,q}(X, E).$$

Connecting Harmonic theory and Dolbeault cohomology

Define:

$$\Phi : \text{Ker}(\varepsilon^{p;q}(X, E) \xrightarrow{\overline{\partial}} \varepsilon^{p;q+1}(X, E)) \to \mathcal{H}^{p,q}(X, E) \text{ by } \Phi(f) = f_h$$

where f_h is the harmonic part of f. By the first isomorphism theorem, we have:

$$\frac{\text{Ker}(\varepsilon^{p;q}(X, E) \xrightarrow{\overline{\partial}} \varepsilon^{p;q+1}(X, E))}{\text{Ker}(\Phi)} \cong \text{Im}(\Phi).$$

Properties of $\overline{\square}$ show that Φ is surjective, and $\text{Ker}(\Phi) = \text{Im}(\varepsilon^{p;q-1}(X, E) \xrightarrow{\overline{\partial}_E} \varepsilon^{p;q}(X, E))$. Hence we have:

Result 2

$\mathcal{H}^{p,q}(X, E) \cong H^{p,q}(X, E)$. Perhaps the most important ingredient of the Serre duality!

The Serre Duality

The Serre Duality unifies all the constructions above and provides a canonical relation between specific cohomology groups over a bundle and its dual. More specifically,

Statement

Let $\pi : E \to X$ be a holomorphic bundle over a compact complex manifold X, and let E' be the dual bundle. Then, $H^q(X, \Omega^p(E)) \cong H^{n-q}(X, \Omega^{n-p}(E'))$.

Proof. Results 1, 2, the projection onto Harmonic forms and Dolbeault's theorem give us:

$$\begin{array}{ccc} \varepsilon^{p;q}(X, E) & \xrightarrow{\overline{*}_E} & \varepsilon^{n-p;n-q}(X, E') \\ \downarrow H_{\overline{\square}} & & \downarrow H_{\overline{\square}} \\ \mathcal{H}^{p,q}(X, E) & \xrightarrow{\overline{*}_E} & \mathcal{H}^{n-p,n-q}(X, E') \\ \downarrow \cong & & \downarrow \cong \\ H^{p,q}(X, E) & & H^{n-p,n-q}(X, E') \\ \downarrow \cong & & \downarrow \cong \\ H^q(X, \Omega^p(E)) & & H^{n-q}(X, \Omega^{n-p}(E')). \end{array}$$

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