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On Choquet's Theorem

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Introduction

Choquet's theorem is a result for some compact convex sets in metrizable spaces that provides a method to represent points as 'barycentres' of probability measures supported by its extreme points. It generalises the Krein-Milman theorem in a setting of integrals over some measures, and provides a translation of some geometric properties into measure theoretical analytical ones. In this sense it is a measure theoretic version of the Krein-Milman theorem, or an *integral representation* of the Krein-Milman theorem.

Preliminaries

Many results from topology, functional analysis and measure theory are assumed, but most important are the Hahn-Banach separation theorem, and the Riesz representation theorem.

Theorem 1: Hahn-Banach Separation

Let X be a topological vector space on which X^* separates points. Suppose $A \subseteq X$ and $B \subseteq X$ are disjoint nonempty compact convex sets in X. Then there exists $\Lambda \in X^*$ such that

$$\sup_{x \in A} \operatorname{Re}(\Lambda x) < \inf_{x \in B} \operatorname{Re}(\Lambda y)$$

for every $x \in A,\ y \in B$

Theorem 2: Riesz Representation

If X is a locally compact Hausdorff space, then every bounded linear functional Φ on $C_0(X)$ is represented by a unique regular complex Borel measure μ, in the sense that

$$\Phi f = \int_X f \, d\mu$$

for every $f \in C_0(X)$. Moreover $\|\Phi\| = |\mu|(X)$.

Definition: Extreme Set

Let X be a vector space and $K \subseteq X$. A set $\varnothing \neq S \subseteq K$ is said to be an *extreme set* of K if for every $x \in K$, $y \in K$ and $t \in (0, 1)$ where

$$(1 - t)x + ty \in S$$

we have $x \in S$ and $y \in S$.

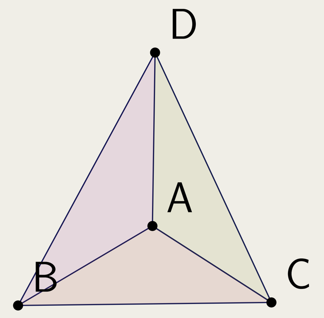
Theorem 3: Carathéodory

If $X \subseteq \mathbb{R}^n$ is compact convex, then every $x \in X$ is a convex combination of at most $n + 1$ extreme points of X.

Finite Dimensional Simplex

The classical result of Carathéodory is an example of what we are trying to generalise into infinite dimensions. It states that any point in the convex hull (denoted co) of a subset of $\mathbb{R}^n$ lies in an n-simplex formed by the points of the set. Here, an n-**simplex** is the convex hull of $n + 1$ points that are affinely independent.

Figure 1. Finite Dimensional Simplex in $\mathbb{R}^3$



We wish to generalise this to infinite dimensions, which yields the Krein-Milman theorem.

Theorem 4: Krein-Milman

Let X be a topological vector space on which X^* separates points. If $\varnothing \neq K \subseteq X$ is a compact convex set, then $K = \overline{\operatorname{co}}(E(K))$.

Vector-valued Integration

First, we will define 'vector-valued' integration. Let Q be a measure space with real or complex measure μ. Let $f : Q \to X$ be a function where X is a topological vector space. We wish to associate a value to

$$\int_Q f \, d\mu$$

Definition: Vector-valued Integration

Let Q be a measure space and μ be a measure (real or complex) on it. Let X be a topological vector space on which X^* separates points. Let $f : Q \to X$ be a function such that Λf is integrable for every $\Lambda \in X^*$. Note that if $q \in Q$ then $(\Lambda f)(q) = \Lambda(f(q))$. If there exists $y \in X$ such that

$$\Lambda y = \int_Q (\Lambda f) \, d\mu$$

for each $\Lambda \in X^*$ then we define

$$\int_Q f \, d\mu = y$$

Theorem 5

Let X be a topological vector space on which X^* separates points and let μ be a Borel probability measure on a compact Hausdorff space Q. Then if $f : Q \to X$ is continuous and $\overline{\operatorname{co}}(f(Q))$ is compact, then

$$y = \int_Q f \, d\mu$$

exists.

Integral Representations

Our aim is to generalise the Krein-Milman theorem in a theory of representation of points by measures on the space. We first note some theorems which give points as barycentres of probability measures that represent them. We wish to extend this theory to a case that matches that of the Krein-Milman theorem, and we wish to sharpen it once we do.
What we mean by 'representing' is denoted by the subsequent definition.

Definition: Barycentre

Let Q be a compact Hausdorff subset of a locally convex topological vector space X. x is said to be the barycentre of a positive regular Borel measure μ on Q if

$$\Lambda x = \int_Q \Lambda \, d\mu \ \ \forall \Lambda \in X^*$$

This is the sense in which we say that x is **represented** by μ.

Theorem 6

Suppose that Q a compact subset of a locally convex topological vector space X and $\overline{H} = \overline{\operatorname{co}}(Q)$. Then a point $y \in \overline{H}$ iff there is a probability measure μ on Q such that μ represents y. In other words

$$\int_Q \Lambda \, d\mu = \Lambda y \ \ \forall \Lambda \in X^*$$

Theorem 7: Generalised Krein-Milman

Let X be a locally convex topological vector space, and let $Q \subseteq X$ be compact convex. Then every $x \in Q$ is the barycentre of a probability measure μ on Q supported on $\overline{E(Q)}$.

Theorem 8: Bauer

Let X be a locally convex topological vector space and Q be a compact convex subset. Then $x \in E(Q)$ iff δ_x, the Dirac measure at x, is the only probability measure on Q whose barycentre is x.

Choquet's Theorem

We are able to connect the branches of convexity theory, measure theory and functional analysis with this theorem of representation that is in a sense a generalisation of the kinds of theorems as Carathéodory, Krein-Milman, etc.

Theorem 9: Choquet

Suppose Q is a metrisable compact convex subset of a locally convex topological vector space X. Let $x_0 \in Q$. Then there is a probability measure $\mu \in \mathscr{P}(Q)$ whose barycentre is x_0, supported by $E(Q)$:

$$\Lambda x_0 = \int_{E(Q)} \Lambda \, d\mu \ \ \forall \Lambda \in X^*$$

Applications

Choquet theory has numerous applications in mathematical physics, economics, optimisation, and areas in pure mathematics. One such recent application from mathematical finance is explained from [Delbaen 2024], that demonstrates the usefulness and wide application of the theorem. Note that a space X is said to be *Polish* if it is separable, metrisable, and complete as a metric space.

Concave Monetary Utility Function

Let X be a Polish space, and $u : C_b(X) \to \mathbb{R}$ be a function. We call u a concave utility monetary function if

1. u is a concave function
2. for every $f \in C_b(X)$ such that f is nonnegative, we have $u(f)$ to be nonnegative
3. u is monetary, i.e. for every $a \in \mathbb{R}$, there is $u(f + a) = u(f) + a$ for every $f \in C_b(X)$
4. u is positively homogeneous, i.e., for every $f \in C_b(X)$ and $\lambda \geq 0$, we have $u(\lambda f) = \lambda u(f)$
5. u satisfies the **Fatou property**, i.e., for every sequence of functions $\{f_n\}_{n \geq 1} \subseteq C_b(X)$ such that $f_n(x) \to f(x)$ for every $x \in X$, and f_n is decreasing, we have $u(f_n) \to u(f)$ and $u(f_n)$ is decreasing, for $f \in C_b(X)$.

Theorem 10: (Delbaen)

Suppose X is a Polish space and $u : C(X) \to \mathbb{R}$ is a function such that

1. *u is a concave and positively homogenous function*
2. *u is a monetary function*
3. *u is monotone*

Then there exists a compact set $L \subseteq X$, and a convex set $\mathscr{S} \subseteq X$ of probability measures (positive regular Borel measures of total mass 1) with support contained in L, i.e. $\mu \in \mathscr{S} \implies \mu(L) = 1$ such that for every $f \in C(X)$, $u(f) = \inf\{\mu(f) : \mu \in \mathscr{S}\}$

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